The MP Algorithm

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Outline

- The MP and MMP algorithm

- Unreduced lower Hessenberg matrices

The Minimal Polynomial

Given a matrix $A \in M_n(\mathbb{C})$, the minimal polynomial of A , denoted by $q_A(t)$, is defined to be the monic polynomial of minimal degree such that $q_A(A) = 0$.

The MP algorithm

Our algorithm depends on the set $D = \left\{A^0, A, A^2, \ldots, A^n\right\}$ being linearly dependent.

The $rvec$ operation

Let $A = [a_{ij}] \in M_n(\mathbb{C})$. Then a row vector, denoted by **rvec(A)**, is defined by

$$rvec(A) = [a_{11}, \ldots, a_{1n} : a_{21}, \ldots, a_{2n} : \ldots : a_{n1}, \ldots, a_{nn}] \in \mathbb{C}_{n^2}.$$

1. The map $f : M_n(\mathbb{C}) \to \mathbb{C}_{n^2}$ be defined by $f(A) = rvec(A)$ is an isomorphism.

2. The set

$$S = \{v_0 = f(I), v_1 = f(A), v_2 = f(A^2), \ldots, v_n = f(A^n)\}$$

is linearly dependent.

3. Why transform a matrix into a vector?

The GU matrix

Let $\{v_0, \ldots, v_k\} \subseteq \mathbb{C}_{n^2}$ and $B_{k+1} \in M_{k+1}(\mathbb{C})$ be given. We define $G_{\{v_0,\ldots,v_k\}}(B_{k+1}) \in M_{k+1,n^2+k+1}(\mathbb{C})$ to be the matrix,

$$G_{\{v_0,\ldots,v_k\}}(B_{k+1}) \equiv \left[\begin{array}{c|c} v_0 & \\ v_1 & \\ \vdots & B_{k+1} \\ v_k & \end{array}\right].$$

The matrix $G_{\{v_0,\ldots,v_k\}}(B_{k+1})$ will be called the **Gaussian updating (GU) matrix.**

How the GU matrix is used in the MP algorithm

1. Begin with $G_{\{v_0\}}(B_1) = [v_0 \| 1] = [e_1^T \quad e_2^T \quad \cdots \quad e_n^T \quad \| \quad 1]$ where $B_1 = I_1$

2. Next, construct $G_{\{v_0,v_1\}}(B_2)$ where $B_2 = \begin{bmatrix} B_1 & 0 \\ 0 & 1 \end{bmatrix} \in M_2(\mathbb{C})$.

3. Gaussian row operations are performed to determine whether or not v_0 and v_1 are linearly dependent.

4. Out of this we have a new GU matrix, $G_{\{v_0,v_1'\}}(B_2')$, where v_1' is the vector obtained from v_1 in the Gaussian elimina-

tion, and B_2' is the matrix obtained from B_2 by the Gaussian elimination.

5. Successively, we construct the GU matrix $G_{\left\{v_0, v_1', \ldots, v_{k-1}', v_k\right\}}(B_{k+1})$, where $B_{k+1} = \begin{bmatrix} B_k' & 0 \\ 0 & e_k^T \end{bmatrix}$.

6. Gaussian elimination is used to determine whether or not the newly introduced vector, v_k, is linearly dependent to the vectors in the set $\left\{v_0, v_1', \ldots, v_{k-1}'\right\}$.

7. From this we obtain a new GU matrix, $G_{\left\{v_0, v_1', \ldots, v_{k-1}', v_k'\right\}}(B_{k+1}')$, where v_k' is the vector obtained from v_k in the Gaussian elim-

ination, and B'_{k+1} is the matrix obtained from B_{k+1} in the Gaussian elimination.

8. This process must produce a zero vector, v'_k, for some $k \leq n$

The matrix B_{k+1} in the GU matrix

1. Start with $B_1 = I_1$

2. $B_2 = \begin{bmatrix} B_1 & 0 \\ 0 & e_1^T \end{bmatrix}$

3. Construct the GU matrix $G_{\{v_0,v_1\}}(B_2)$

4. Use Gaussian elimination to obtain the GU matrix $G_{\{v_0,v_1'\}}(B_2')$

5. The first column of B_2' is the coefficient of I_n in the linear combination of v_1' and the second column of B_2' is the coefficient of A in the linear combination of v_1'.

6. When $G_{\{v_0,v'_1,\ldots,v'_k\}}(B'_{k+1})$ is obtained from $G_{\left\{v_0,v'_1\ldots,v'_{k-1},v_k\right\}}(B_{k+1})$ using Gaussian elimination.

   (a) Then the first column of B'_{k+1} is the coefficient of I_n in the linear combination of v'_k;

   (b) The second column of the matrix B'_{k+1} is the coefficient of A in the linear combination of v'_k;

   (c) In general the k-th column of the matrix B'_{k+1} is the coefficient of A^k in the linear combination of v'_k.

7. The coefficients of the minimal polynomial are given in the last row of B'_{k+1}.

The Minimal Polynomial Algorithm (MP)

For a given $A \in M_n(\mathbb{C})$, let $v_i = rvec(A^i)$, and do the following.

Step 1. (Initialization). Create $G_{\{v_0\}}(I_1)$, set $v_0 \equiv v_0'$, $i = 1$, and $B_1 \equiv I_1$.

Step 2. Compute v_i and construct $G_{\left\{v_0', \ldots, v_{i-1}', v_i\right\}}(B_{i+1})$ where $B_{i+1} \equiv \begin{bmatrix} B_i' & 0 \\ 0 & e_i^T \end{bmatrix}$. Use Gaussian elimination to obtain $G_{\left\{v_0', \ldots, v_{i-1}', v_i'\right\}}(B_{i+1}')$.

- If $v_i' \equiv 0$ stop and proceed to Step 3.

- If $v_i' \neq 0$, increment i by 1 and repeat Step 2.

Step 3. For $i = k$ such that $v_k' \equiv 0$, the entries of the last row of B_{k+1}', $b_{k+1,j} \in \mathbb{C}$ with $j = 1, \ldots, k+1$, are the coefficients of the minimal polynomial of the matrix $A \in M_n(\mathbb{C})$.

Example 1 We use the MP algorithm to compute the minimal polynomial of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix} \in M_3(\mathbb{R})$$

1. Compute $v_0 = v'_0 = rvec(I_3) = \left[\begin{array}{ccccccccccc} 1 & 0 & 0 & : & 0 & 1 & 0 & : & 0 & 0 & 1 \end{array}\right]$

2. Construct

$$G_{\{v'_0\}}(I_1) = \left[\begin{array}{ccccccccccccc} 1 & 0 & 0 & : & 0 & 1 & 0 & : & 0 & 0 & 1 & \| & 1 \end{array}\right].$$

Compute $v_1 = rvec(A) = \left[\begin{array}{ccccccccccc} 1 & 1 & 0 & : & -1 & 2 & 1 & : & 2 & 0 & 1 \end{array}\right]$

Construct

$$G_{\{v_0',v_1\}}(I_2) = \left[\begin{array}{cccccccccccccc} 1 & 0 & 0 & : & 0 & 1 & 0 & : & 0 & 0 & 1 & \| & 1 & 0 \\ 1 & 1 & 0 & : & -1 & 2 & 1 & : & 2 & 0 & 1 & \| & 0 & 1 \end{array}\right].$$

Obtain the GU matrix by Gaussian elimination

$$G_{\{v'_0,v'_1\}}(B'_2) = \left[\begin{array}{ccccccccccccccc} 1 & 0 & 0 & : & 0 & 1 & 0 & : & 0 & 0 & 1 & \| & 1 & 0 \\ 0 & 1 & 0 & : & -1 & 1 & 1 & : & 2 & 0 & 0 & \| & -1 & 1 \end{array}\right].$$

Since $v'_1 \neq 0$, the alorithm continues

Compute

$$v_2 = rvec(A^2) = \left[\begin{array}{ccccccccccc} 0 & 3 & 1 & : & -1 & 3 & 3 & : & 4 & 2 & 1 \end{array}\right]$$

Consruct the GU matrix

$$G_{\{v_0', v_1', v_2\}}(B_3) = \left[\begin{array}{ccccccccccccccc} 1 & 0 & 0 & : & 0 & 1 & 0 & : & 0 & 0 & 1 & \| & 1 & 0 & 0 \\ 0 & 1 & 0 & : & -1 & 1 & 1 & : & 2 & 0 & 0 & \| & -1 & 1 & 0 \\ 0 & 3 & 1 & : & -1 & 3 & 3 & : & 4 & 2 & 1 & \| & 0 & 0 & 1 \end{array}\right].$$

Obtain the GU matrix by Gaussian elimination

$$G_{\{v'_0, v'_1, v'_2\}}(B'_3) = \left[\begin{array}{ccccccccccccccc} 1 & 0 & 0 & : & 0 & 1 & 0 & : & 0 & 0 & 1 & \| & 1 & 0 & 0 \\ 0 & 1 & 0 & : & -1 & 1 & 1 & : & 2 & 0 & 0 & \| & -1 & 1 & 0 \\ 0 & 0 & 1 & : & 2 & 0 & 0 & : & -2 & 2 & 1 & \| & 3 & -3 & 1 \end{array} \right].$$

Since $v'_2 \neq 0$, the algorithm continues.

Compute

$$v_3 = rvec(A^3) = \left[\begin{array}{ccccccccccc} -1 & 6 & 4 & : & 2 & 5 & 6 & : & 4 & 8 & 3 \end{array}\right]$$

Construct the GU matrix $G_{\{v'_0, v'_1, v'_2, v_3\}}(B_4) =$

$$\left[\begin{array}{ccccccccccccccc} 1 & 0 & 0 & : & 0 & 1 & 0 & : & 0 & 0 & 1 & \| & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & : & -1 & 1 & 1 & : & 2 & 0 & 0 & \| & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & : & 2 & 0 & 0 & : & -2 & 2 & 1 & \| & 3 & -3 & 1 & 0 \\ -1 & 6 & 4 & : & 2 & 5 & 6 & : & 4 & 8 & 3 & \| & 0 & 0 & 0 & 1 \end{array}\right].$$

The new GU matrix is $G_{\{v_0',v_1',v_2',v_3'\}}(B_4') =$

$$\left[\begin{array}{ccccccccccccccccc} 1 & 0 & 0 & : & 0 & 1 & 0 & : & 0 & 0 & 1 & \| & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & : & -1 & 1 & 1 & : & 2 & 0 & 0 & \| & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & : & 2 & 0 & 0 & : & -2 & 2 & 1 & \| & 3 & -3 & 1 & 0 \\ 0 & 0 & 0 & : & 0 & 0 & 0 & : & 0 & 0 & 0 & \| & -5 & 6 & -4 & 1 \end{array}\right].$$

Since $v_3' \equiv 0$, the algorithm terminates.

The coefficients of the minimal polynomial can be read off from the last row of the matrix

$$B'_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ -5 & 6 & -4 & 1 \end{bmatrix}.$$

The minimal polynomial of A is $q_A(t) = -5 + 6t - 4t^2 + t^3$

The MMP algorithm

The Modified Minimal Polynomial Algorithm (MMP)

For a given $A \in M_n(\mathbb{C})$, do the following.

Step 1. Create $G_{\{v_0\}}(I_1)$, set $v_0 = rvec(I_n) \equiv v'_0$, set $i = 1$, and $B_1 \equiv I_1$

Step 2. Compute $v_i = v'_{i-1}(I \otimes A)$ and construct the GU matrix $G_{\left\{v'_0,\ldots,v'_{i-1},v_i\right\}}(B_{i+1})$ where $B_{i+1} \equiv \begin{bmatrix} B'_i & 0 \\ 0 & b \end{bmatrix}$, such that $b \in \mathbb{C}_i$ are the entries of the last row of B'_i. Use Gaussian elimination to obtain $G_{\left\{v'_0,\ldots,v'_{i-1},v'_i\right\}}(B'_{i+1})$.

- If $v'_i \equiv 0$ stop and proceed to Step 3.
- If $v'_i \neq 0$, increment i by 1 and repeat Step 2.

Step 3. For $i = k$ such that $v'_k \equiv 0$, the entries of the last row of B'_{k+1}, $b_{k+1,j} \in \mathbb{C}$ for $j = 1, \ldots, k+1$, are the coefficients of the minimal polynomial of the matrix $A \in M_n(\mathbb{C})$.

The benefits

1. v_k is obtained from v'_{k-1} by $v_i = v'_{i-1}(I \otimes A)$

2. Obtaining B_{k+1}

The matrix B_{k+1}

Suppose we have computed the following GU matrix

$$G_{\left\{v'_0,\ldots,v'_{k-1}\right\}}(B'_k) = \left[\begin{array}{c|cccc} v'_0 & \| & & & \\ v'_1 & \| & & B'_{k-1} & \\ \vdots & \| & & & \\ v'_{k-1} & \| b_1 & & \ldots & b_k \end{array}\right].$$

The $(k+1)$-th row of the matrix B_{k+1} may be obtained by shifting the entries to the right 1 entry.

$$G_{\left\{v'_0,\ldots,v'_{k-1},v_k\right\}}(B_{k+1}) = \begin{bmatrix} v'_0 & \| & & & \\ v'_1 & \| & B'_{k-1} & & \\ \vdots & \| & & & \\ v'_{k-1} & \|b_1 & \ldots & b_k & 0 \\ v_k & \|0 & b_1 & \ldots & b_k \end{bmatrix}.$$

Example 2

We use the MMP algorithm to compute the minimal polynomial of the matrix

$$A = \begin{bmatrix} 3 & -1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \in M_4(\mathbb{C}).$$

Compute

$$v_0' = \left[\ 1 \quad 0 \quad 0 \quad 0 \quad : 0 \quad 1 \quad 0 \quad 0 \quad : \quad 0 \quad 0 \quad 1 \quad 0 : \quad 0 \quad 0 \quad 0 \quad 1\ \right]$$

Construct

$$G_{\{v_0'\}}(I_1) = [\, 1\ 0\ 0\ 0\ :\ 1\ 0\ 0\ 0\ :\ 0\ 0\ 1\ 0\ :\ 0\ 0\ 0\ 1\ \|\ 1\,]$$

Compute

$$v_1 = v_0'(I \otimes A)$$

$$= \left[\begin{array}{cccccccccccccccc} 3 & -1 & -1 & 0 & : 1 & 1 & -1 & 0 & : & 1 & -1 & 1 & 0 & : 1 & -1 & 0 & 1 \end{array}\right]$$

Construct

$$G_{\{v_0', v_1\}}(I_2) = \left[\begin{array}{c} 1 \;\; 0 \;\; 0 \;\; 0 : 0 \;\; 1 \;\; 0 \;\; 0 : 0 \;\; 0 \;\; 1 \;\; 0 : 0 \;\; 0 \;\; 0 \;\; 1 \;\|\; 1 \;\; 0 \\ 3 \;\; -1 \;\; -1 \;\; 0 : 1 \;\; 1 \;\; -1 \;\; 0 : 1 \;\; -1 \;\; 1 \;\; 0 : 1 \;\; -1 \;\; 0 \;\; 1 \;\|\; 0 \;\; 1 \end{array}\right]$$

Obtain the GU matrix by Gaussian elimination

$$G_{\{v'_0,v'_1\}}(B'_2) = \left[\begin{array}{cccccccccccccccccccccc} 1 & 0 & 0 & 0 & : & 0 & 1 & 0 & 0 & : & 0 & 0 & 1 & 0 & : & 0 & 0 & 0 & 1 & \| & 1 & 0 \\ 0 & -1 & -1 & 0 & : & 1 & -2 & -1 & 0 & : & 1 & -1 & -2 & 0 & : & 1 & -1 & 0 & -2 & \| & -3 & 1 \end{array}\right]$$

Since $v'_1 \neq 0$, we continue the algorithm

Compute

$$v_2 = v_1'(I \otimes A)$$

$$= \left[\begin{array}{cccc} -2e_1^T & -2e_2^T & -2e_3^T & -2e_4^T \end{array} \right]$$

Construct $G_{\{v_0',v_1',v_2\}}(B_3) =$

$$\left[\begin{array}{cccccccccccccccccccc} 1 & 0 & 0 & 0 & : & 0 & 1 & 0 & 0 & : & 0 & 0 & 1 & 0 & : & 0 & 0 & 0 & 1 & \| & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & : & 1 & -2 & -1 & 0 & : & 1 & -1 & -2 & 0 & : & 1 & -1 & 0 & -2 & \| & -3 & 1 & 0 \\ -2 & 0 & 0 & 0 & : & 0 & -2 & 0 & 0 & : & 0 & 0 & -2 & 0 & : & 0 & 0 & 0 & -2 & \| & 0 & -3 & 1 \end{array} \right]$$

Obtain the GU matrix by Gaussian elimination $G_{\{v'_0,v'_1,v'_2\}}(B'_3) =$

$$\left[\begin{array}{cccccccccccccccccccccc} 1 & 0 & 0 & 0 & : & 0 & 1 & 0 & 0 & : & 0 & 0 & 1 & 0 & : & 0 & 0 & 0 & 1 & \| & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & : & 1 & -2 & -1 & 0 & : & 1 & -1 & -2 & 0 & : & 1 & -1 & 0 & -2 & \| & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & : & 0 & 0 & 0 & 0 & : & 0 & 0 & 0 & 0 & : & 0 & 0 & 0 & 0 & \| & 2 & -3 & 1 \end{array}\right]$$

Since $v'_2 \equiv 0$ the MMP algorithm terminates.

The minimal polynomial is $q_A(t) = 2 - 3t + t^2$.

The MMP Algorithm for Unreduced Lower Hessenberg Matrices

For an unreduced lower Hessenberg matrix $A \in M_n(\mathbb{C})$, let v_i be the first row of the matrix $A^{(i)}$, $A^{(0)} = I_n$, and do the following.

Step 1. Create $G_{\{v_0\}}(I_1)$, set $v_0 = e_1^T \equiv v_0'$, where $e_1^T \in \mathbb{C}_n$, set $i = 1$, and $B_1 \equiv I_1$.

Step 2. Compute $v_i = v_{i-1}' A$ and construct the GU matrix $G_{\left\{v_0', \ldots, v_{i-1}', v_i\right\}}(B_{i+1})$ where $B_{i+1} \equiv \begin{bmatrix} B_i' & 0 \\ 0 & b \end{bmatrix}$, such that $b \in \mathbb{C}_i$ are the entries of the last row of B_i'. Use Gaussian elimination to obtain $G_{\left\{v_0', \ldots, v_{i-1}', v_i'\right\}}(B_{i+1}')$.

- If $v_i' \equiv 0$ stop and proceed to Step 3.
- If $v_i' \neq 0$, increment i by 1 and repeat Step 2.

Step 3. For $i = k$ such that $v_k' \equiv 0$, the entries of the last row of B_{k+1}', $b_{k+1,j} \in \mathbb{C}$ with $j = 1, \ldots, k+1$, are the coefficients of the minimal polynomial of the matrix $A \in M_n(\mathbb{C})$.

Benefit

Only need to use the first row of the matrix.

Example 5

Consider the lower Hessenberg matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \in M_3(\mathbb{R}).$$

Start with $v'_0 = [1\,0\,0]$.

Construct the GU matrix

$$G_{\{v'_0\}}(I_1) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \end{array}\right].$$

Compute $v_1 = v_0' A = \left[\begin{array}{ccc} 1 & 1 & 0 \end{array} \right]$.

Construct the GU matrix

$$G_{\{v_0', v_1\}}(I_2) = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{array} \right].$$

Obtain the GU matrix by Gaussian elimination

$$G_{\{v'_0,v'_1\}}(B'_2) = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \end{array}\right].$$

Since $v'_1 \neq 0$, the algorithm continues.

Compute $v_2 = v_1' A = \left[\begin{array}{ccc} 2 & 1 & 1 \end{array} \right]$.

Construct the GU matrix

$$G_{\{v_0', v_1', v_2\}}(B_3) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 2 & 1 & 1 & 0 & -1 & 1 \end{array} \right].$$

Obtain the GU matrix by Gaussian elimination

$$G_{\{v_0',v_1',v_2'\}}(B_3') = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{array}\right].$$

Since $v_2' \neq 0$, the algorithm continues.

Compute $v_3 = v_2'A = \left[\begin{array}{ccc} 1 & 2 & 3 \end{array}\right]$.

Construct the GU matrix

$$G_{\{v_0',v_1',v_2',v_3\}}(B_4) = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -2 & 1 & 0 \\ 1 & 2 & 3 & 0 & -1 & -2 & 1 \end{array}\right].$$

Obtain the GU matrix by Gaussian elimination

$$G_{\{v'_0,v'_1,v'_2,v'_3\}}(B'_4) = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 4 & 3 & -5 & 1 \end{array}\right].$$

Since $v'_3 \equiv 0$ the algorithm terminates.

The minimal polynomial of the matrix A is $q_A(t) = 4 + 3t - 5t^2 + t^3$.

THANK YOU